Numerical Solutions of PDE's

(We'll barely scratch the surface here)

We'll work with the heat equation:

$$u_{xx} + u_{yy} + u_{zz} = u_t$$

(here, the solution is $u(x,y,z,t)$ representing the temperature at coordinates x,y,z & time t)

We'll also need some boundary conditions to have any hope of uniqueness of solutions

For simplicity let's consider the 1D heat equation with boundary conditions:

$$u_{xx} = u_t$$

$u(x,0) = g(t)$ → temp everywhere at $t=0$

$u(0,t) = a(t)$ → temp at $x=0$ at all time

$u(1,t) = b(t)$ → temp at $x=1$ at all time.

→ models the temp of a rod of length 1.

Goal: Find $u(x,t)$.

Numerical Method: Finite Differences

Idea: discretize both t & x coordinates & approximate the derivatives using, e.g., $f'(x) = [f(x+h) - f(x)]/h$

$f''(x) = [f(x+h) - 2f(x) + f(x-h)]/h^2$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}(x,t)$$

$$\approx \frac{1}{h^2}\left[u(x+h,t) - 2u(x) + u(x-h,t)\right]$$

$$u_t = \frac{\partial u}{\partial t}(x,t) \approx \frac{1}{k}\left[u(x,t+k) - u(x,t)\right]$$

Let's discretize:

$v(x,t)$ = numerical sol'n, satisfies

$$\frac{1}{h^2}\left[v(x_{i+1}, t_j) - 2v(x_i, t_j) + v(x_{i-1}, t_j)\right]$$
$$= \frac{1}{k}\left[v(x_i, t_{j+1}) - v(x_i, t_j)\right]$$

where $x_{i+1} = x_0 + (i+1)h$
$t_{j+1} = t_0 + (j+1)k$

(mesh points, n+2 of them)

Simplify notation:

$$\frac{1}{h^2}\left[v_{i+1,j} - 2v_{ij} + v_{i-1,j}\right] = \frac{1}{k}\left[v_{i,j+1} - v_{ij}\right]$$

Notice: $j=0$ $(t=t_0)$

$\Rightarrow$ we know everything from the bndry conds (except $v_{i,1}$)

$\Rightarrow$ **can calculate $v_{i,1}$**

Similarly: Given $v_{i,j}$, $v_{i-1,j}$, $v_{i+1,j}$ we can find $v_{i,j+1}$

$$v_{i,j+1} = s v_{i-1,j} + (1-2s) v_{ij} + s v_{i+1,j}$$

$s = k/h^2$

In matrix-vector notation:

(here we consider $a(t) = b(t) = 0$)

$$V_j = \begin{bmatrix} v_{1j} \\ v_{2j} \\ \vdots \\ v_{nj} \end{bmatrix} \rightsquigarrow \text{(numerical solution at all mesh points (interior) at time } t_j)$$

$$\Rightarrow V_{j+1} = A V_j$$

$$A = \begin{bmatrix} 1-2s & s & 0 & \cdots & 0 \\ s & 1-2s & s & 0 & \cdots & 0 \\ 0 & s & 1-2s & s & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & & \vdots \\ & \vdots & & & & & s \\ 0 & 0 & \cdots & & s & & 1-2s \end{bmatrix}$$

Stability analysis

$a(t) = b(t) = 0 \Rightarrow$ want

$$\lim_{t \to \infty} (\text{our solution}) = 0$$

(This is a heat equation)

but $V_j = A V_{j-1} = \cdots = A^j V_0$

Linear algebra tells us:

$$\lim_{j\to\infty} A^j V_0 = 0 \iff \rho(A) < 1$$

$\rho(A)$: spectral radius $\left(\max\{|\lambda_1|, \cdots, |\lambda_n|\}\right)$

So, we need to pick $s = (k/h^2)$ so that $\rho(A) < 1$

Fact: The eigenvalues of A are $\lambda_j = 1 - 2s(1-\cos\theta_j)$

$\theta_j \hookrightarrow \frac{j\pi}{n+1}$

$j = 1, \cdots, n$

$\Rightarrow$ need $-1 < 1 - 2s(1-\cos\theta_j) < 1$

$\iff s < (1-\cos\theta_j)^{-1} \quad \forall j \in 1, \cdots, n$

(when $j=n$, $\cos\theta_j = \cos\frac{n\pi}{n+1} \approx -1$)

then $s < \left(1-\cos\frac{n\pi}{n+1}\right)^{-1} \approx \frac{1}{2}$

but $s = \frac{k}{h^2} \Rightarrow$ need $k \leq \frac{h^2}{2}$

k: time step; h: mesh separation

e.g. $h = 10^{-3} \Rightarrow k \leq \frac{10^{-6}}{2} = 5\times10^{-7}$

Possible remedy : Implicit methods

Replace

$$\frac{1}{h^2}\left[v_{i+1,j} - 2v_{ij} + v_{i-1,j}\right] = \frac{1}{k}\left[v_{i,j+1} - v_{ij}\right]$$

with (different approx of deriv.)

$$\frac{1}{h^2}\left[v_{i+1,j} - 2v_{ij} + v_{i-1,j}\right] = \frac{1}{k}\left[v_{ij} - v_{i,j-1}\right]$$

$$\Rightarrow -sV_{i-1,j} + (1+2s)V_{i,j} - sV_{i+1,j} = V_{i,j-1}$$

So now

$$AV_j = V_{j-1}$$

$$\begin{bmatrix} 1+2s & -s & 0 & \cdots & 0 \\ -s & 1+2s & -s & 0 & \cdots \\ 0 & -s & 1+2s & -s & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & -s & 1+2s \end{bmatrix}$$

$$\& \quad V_j = A^{-1}V_{j-1} = A^{-j}V_0$$

$\Rightarrow$ want $\rho(A^{-1}) < 1$

but $\lambda_i(A) = 1 + 2s(1 - \cos\theta_i)$, $\theta_i = \frac{\pi i}{n+1}$, $0 < \theta_i < \pi$

> 1

$\Rightarrow \rho(A^{-1}) < 1$